Introduction aux signaux et systèmes
Lecture 1

One-dimensional closed systems

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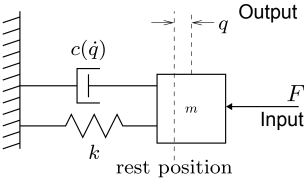
Contents

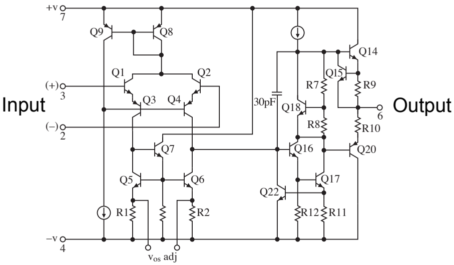
Preliminaries

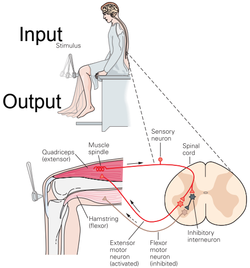
Systems

A "system" is something that receives **input signals** and transforms them into **output signals**.

Examples of systems:







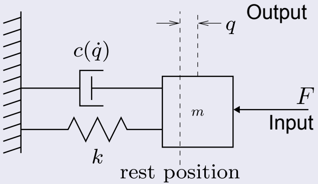
Systems modeling: state-space

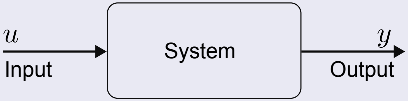
Systems can be represented as "block diagrams", whose input-to-output relationship can be modeled in different ways.

State-space modeling means modeling systems as ordinary differential equations (ODEs) with inputs and outputs: useful when detailed equations can be derived, but sensitive to details.

Systems as block diagrams and their state-space modeling

Spring-mass system





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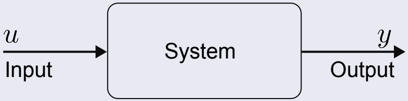
Systems as block diagrams and their state-space modeling

Spring-mass system's state-space model

$$\dot{q} = p$$
$$\dot{p} = -c(p) - k\,q + u$$
$$u = F, \quad y = q$$



State space: $\mathbb{R}^2$

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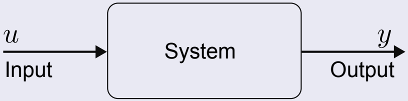
Systems as block diagrams and their state-space modeling

General system's state-space model

$$\begin{aligned} \dot{x} &= f(x,u), \quad x \in \mathbb{R}^n,\, u \in \mathbb{R} \\ y &= g(x,u), \quad y \in \mathbb{R} \end{aligned}$$

State space: $\mathbb{R}^n$



Systems modeling: input-output modeling

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Input-output modeling means modeling systems as operators that map inputs to outputs: more qualitative in nature, but easier to build and manipulate.

Systems as block diagrams and their input-output modeling

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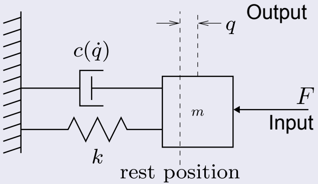

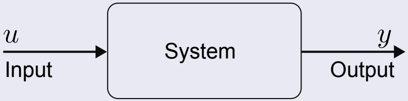

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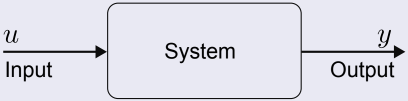
Systems as block diagrams and their input-output modeling

Spring-mass system's transfer function

$$y = \frac{1}{ms^2 + cs + k} u,$$

where $s \in R$ is a frequency variable.



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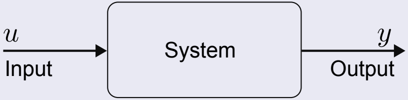
Systems as block diagrams and their input-output modeling

General system's transfer function

$$y = H(s)u,$$

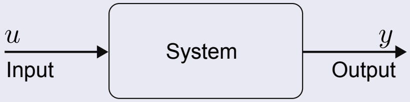
where $s \in R$ is a frequency variable.

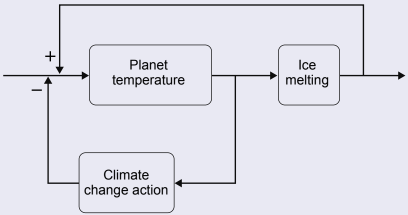


Systems interconnection

The block diagram representation is particular suited to model and analyze complex systems as interconnection of (possibly many) simpler systems.

Complex systems as simple system interconnection





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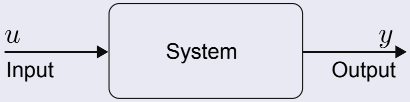
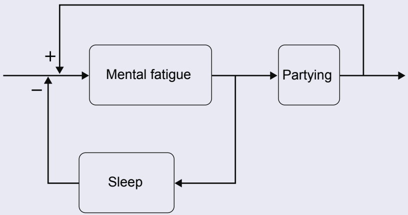
Complex systems as simple system interconnection





From open to closed systems

We will forget for a moment (two lectures) about inputs and outputs and focus on the dynamical behavior of the resulting closed system.

Closing a system

Open system:	Closed system:
$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n,\, u \in \mathbb{R}$ $y = g(x), \quad y \in \mathbb{R}$	$\dot{x} = f(x), \quad x \in \mathbb{R}^n$

This will allows us to introduce many key concepts for systems analysis: **vector fields**, **trajectories**, **equilibrium points**, **stability of equilibria**, **linearization of a system's dynamics**, **linear dynamical systems**.

One-dimensional closed systems

Examples of one-dimensional closed systems

Non-parameterized 1D closed systems $\Longleftarrow$

- $\dot{x} = -x$
- $\dot{x} = \sin(x)$
- $\dot{x} = x^2$
- $\dot{x} = \sqrt{x}$
- $\dot{x} = -x^3 + x$
- $\dot{x} = \frac{e^x}{e^x + e^{-x}}$

Parameterized 1D closed systems

- $\dot{x} = a\,x$
- $\dot{x} = \sin(\omega x + \phi)$
- $\dot{x} = x^2 - t$ (*time-varying$\rightarrow$inputs*)
- $\dot{x} = \sqrt{x - c}$
- $\dot{x} = -x^3 + k\,x$
- $\dot{x} = \frac{e^{x-d}}{e^{x-d} + e^{-x+d}}$

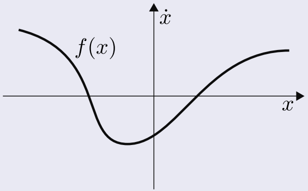
Geometric approach to one-dimensional closed systems

Even in the one-dimensional case $x \in \mathbb{R}$, it is in general impossible to find analytically the solution $x(t)$ to the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0$$

Phase portrait analysis

Study the graph of the **vector field** $\dot{x} = f(x)$ as a function of x on the $(x, \dot{x})$ plane.
The vector field associates to each point x the vector $\dot{x}$.



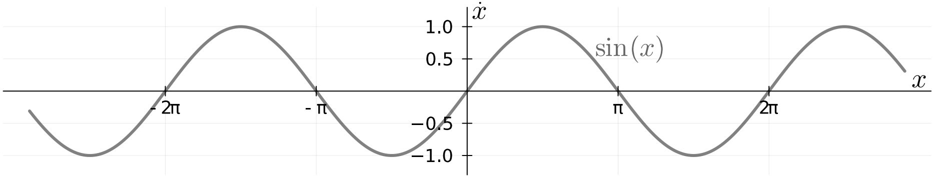
Characterize points in the state space where

- $\dot{x} = f(x) > 0$ ($x(t)$ is increasing)
- $\dot{x} = f(x) < 0$ ($x(t)$ is decreasing)
- $\dot{x} = f(x) = 0$ ($x(t)$ is at rest - or at *equilibrium*)

Phase portrait analysis

We will illustrate the fundamental ideas and methods on the running example[1]

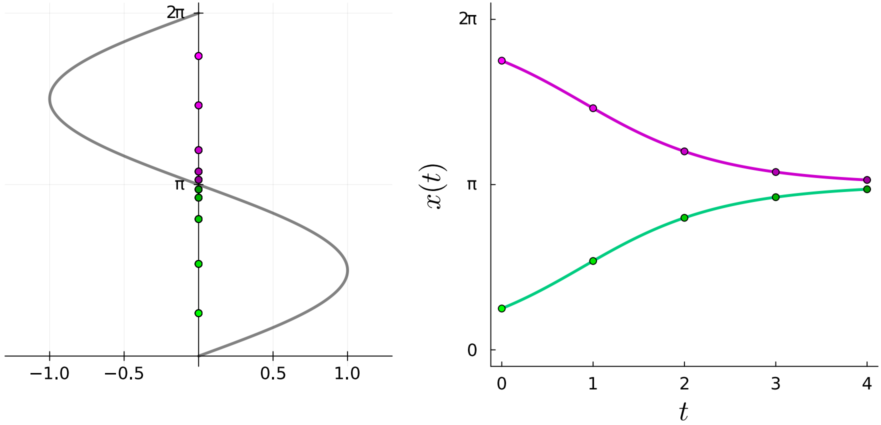
$$\dot{x} = \sin(x), \quad x(0) = x_0$$



[1] For more examples, see Sections 2.2,2.3 of the book Strogatz, S. (1999) Nonlinear dynamics and chaos.

Phase portrait analysis

Hence, at the **qualitative** level, we can fully predict the system's trajectories by studying its phase portrait:



Equilibrium points

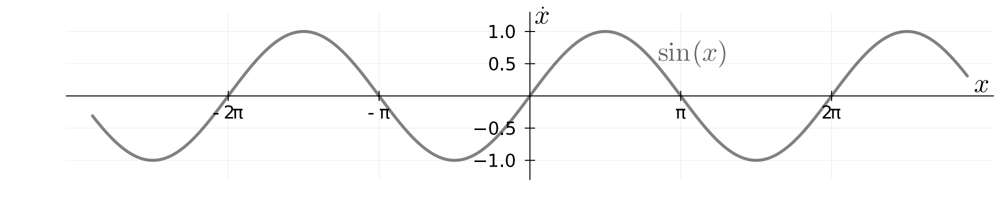
A special kind of points in a system's state space are **equilibrium points** or **equilibria**.

Definition: Equilibrium point

A point $x^* \in \mathbb{R}$ is called an **equilibrium** of the closed system $\dot{x} = f(x)$ if $f(x^*) = 0$.

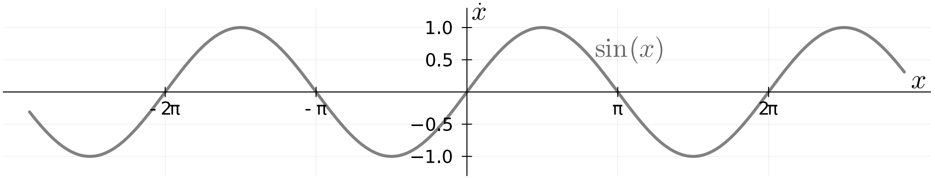
At equilibria, the system is at rest because $x(t) = x^*,\ t \geq 0$, is a solution of $\dot{x} = f(x)$ satisfying $\dot{x}(t) = 0$ for all $t \geq 0$.

Example: $\dot{x} = \sin(x)$



Stability of equilibrium points

We can see that some equilibria “attract” surrounding trajectories, while other “repel” them:



Definition: stable and unstable equilibria

An equilibrium x^* that attract all surrounding trajectories is called **stable**.

An equilibrium that repel all surrounding trajectories is called **unstable**.

(Optional) Rigorous definition of (in)stability

Stable equilibrium

An equilibrium x^* is called **stable** if there exist a neighborhood U of x^* such that for all initial condition $x(0) \in U$ it holds that

$$\lim_{t \to \infty} x(t) = x^*$$

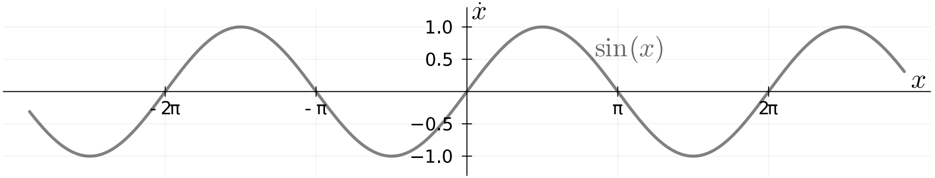
Unstable equilibrium

An equilibrium x^* is called **unstable** if it's not stable. In 1D this is equivalent to the existence of a neighborhood U of x^* such that for all initial condition $x(0) \in U$ it holds that

$$\lim_{t \to -\infty} x(t) = x^*$$

Basins of attraction

For a given stable equilibrium x^* we can ask for which initial conditions does the system trajectory converge to x^*.



Definition: basin of attraction

The **basin of attraction** of a stable equilibrium x^* is the set of all initial conditions x_0 such that $x(t)$ converges to x^* whenever $x(0) = x_0$.

Basins of attractions and gradient systems

By the fundamental theorem of calculus, if the vector field $f(x)$ is an integrable function (e.g., continuous), than in 1D we can always define a primitive for f. For instance, we can let

$$F(x) = \int_0^x f(y)dy\,,$$

in such a way that $f(x) = \frac{dF}{dx}(x)$.

If we let $V(x) = -F(x)$, then

$$\dot{x} = -\frac{dV}{dx}(x)$$

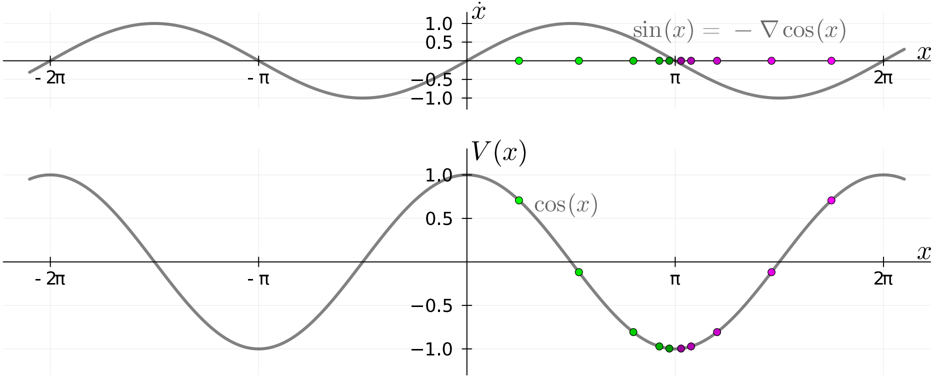
Definition: potential function and gradient system

The function $V(x)$ is called the **potential function** for the **gradient system** $\dot{x} = -\frac{dV}{dx}(x)$.

Basins of attractions and gradient systems

In the gradient system formulation:

- Stable equilibria are minima of the potential function
- Unstable equilibria are maxima of the potential function
- Basins of attractions are the "valleys" in between maxima.



Local stability analysis

The stability of an equilibrium x^* point can also be analyzed analytically using the system's **linearization**. The advantage of the linearization approach is that it is mathematically tractable and generalizes to arbitrary dimensions.

Let $\eta = x - x^*$ be the state relative to the equilibrium x^*. Note that $\eta = 0$ if and only if $x = x^*$ and the closer (further away) x is to (from) x^* the smaller (larger) is η. Furthermore

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x} = f(x) = f(x^* + \eta)$$

Local stability analysis

The stability of x^* can the be characterized as follows

- If for small initial conditions $\eta(0)$ the solution of the system $\dot{\eta} = f(x^* + \eta)$ converges to zero, then x^* is stable: small perturbations to the equilibrium solution decay back to the equilibrium.

- If for small initial conditions $\eta(0)$ the solution of the system $\dot{\eta} = f(x^* + \eta)$ diverges away from zero, then x^* is unstable: small perturbations away from the equilibrium solutions are amplified.

The linearized dynamics

Because we are interested in small η, we can use Taylor's Theorem[2] to expand the function $f(x^* + \eta)$ in η and get

$$\dot{\eta} = \underbrace{f(x^*)}_{\text{Order 0}} + \underbrace{\frac{df}{dx}(x^*)\,\eta}_{\substack{\text{Order 1} \\ \text{(linear)}}} + \underbrace{\mathcal{O}(\eta^2)}_{\text{Higher orders}} .$$

Using the fact that x^* is an equilibrium, hence $f(x^*) = 0$, and that η is small, hence the linear term dominate, we obtain the **linearized dynamics**

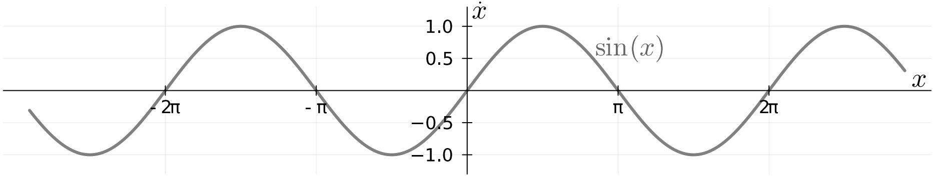
$$\dot{\eta} = \frac{df}{dx}(x^*)\,\eta$$

[2] See, e.g., R.G.Bartle, The Elements of Real Analysis, pag. 211.

The linearized dynamics

Locally around any equilibrium point, the linearized dynamics approximates the vector field by a linear one determined by the slope of the function f at the equilibrium:

$$f(x) \approx \frac{df}{dx}(x^*)(x - x^*)$$

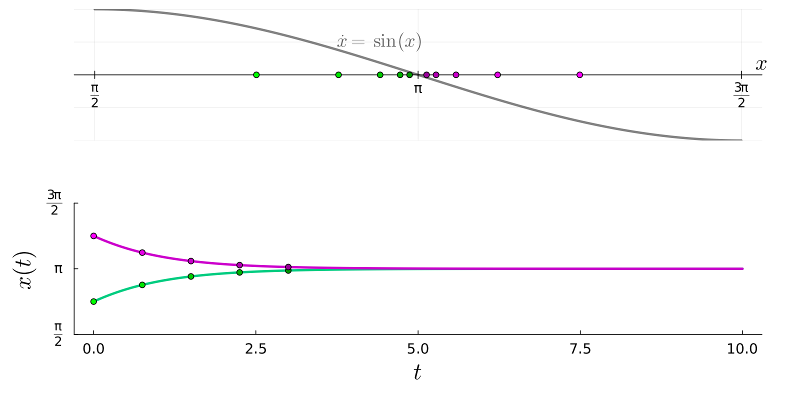


The linearized dynamics

If we let $\lambda = \frac{df}{dx}(x^*)$, the linearized dynamics with initial conditions $\eta(0)$ have solution

$$\eta(t) = \eta(0)e^{\lambda t}$$

Solutions of the linearized dynamics and stability: $\lambda < 0$



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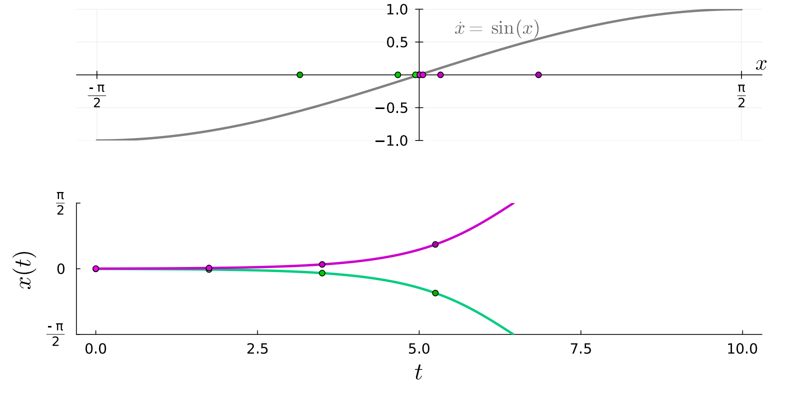
Solutions of the linearized dynamics and stability: $\lambda > 0$



The case of singular linearized dynamics

Definition: Singular equilibrium points

An equilibrium point x^* is called **singular** if $\lambda = \frac{df}{dx}(x^*) = 0$.

Warning: singular equilibria and linearization

It is important to remark that for singular equilibria the system linearization is not informative in general.

$$\dot{x} = x^2 \qquad\qquad \dot{x} = -x^3$$