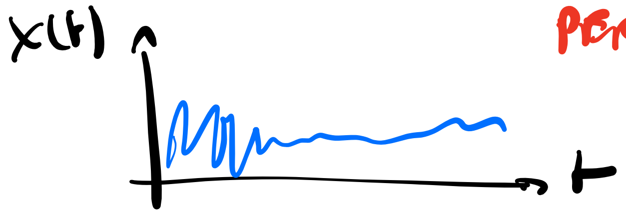
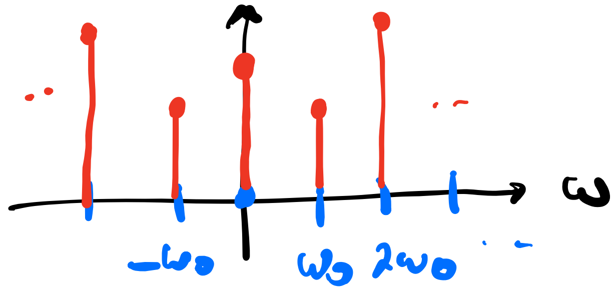
SYST0002 – LECTURE #6 TRANSFORMÉE DE FOURIER.

↳ **RAPPEL** : TEMPOREL ⇄ FRÉQUENTIEL



PÉRIODIQUES T_0

$\omega_0 = \frac{2\pi}{T}$



→ SÉRIES DE FOURIER [CONTINU, PÉRIODIQUE DE PÉRIODE $T_0 = \frac{2\pi}{\omega_0}$]

$$\begin{cases} x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jk\omega_0 t} \\ a_k = \frac{1}{T} \int_T x(t) \cdot e^{-jk\omega_0 t} \end{cases}$$

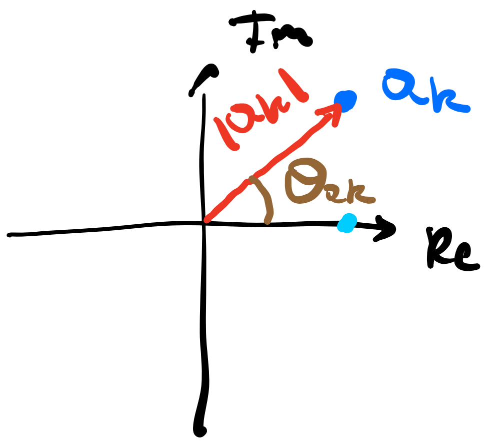
TEMPOREL ⇄ FRÉQUENTIEL

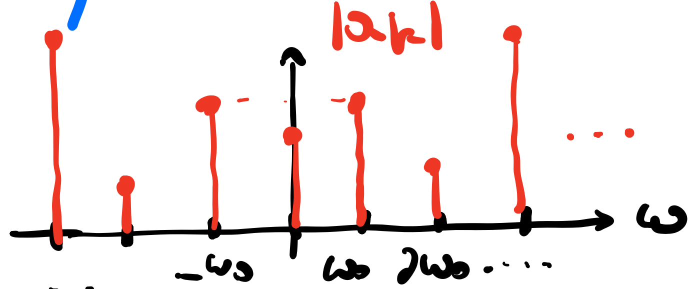
↳ a_k : COEF. DE FOURIER

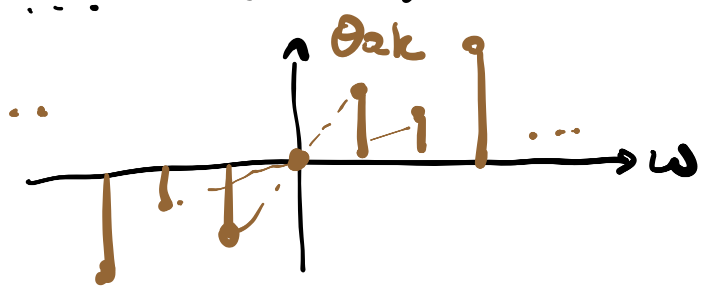
CONTRIBUTION DE CHAQUE HARMONIQUE

- EN AMPLITUDE
- EN PHASE

→ a_k : COMPLEXES





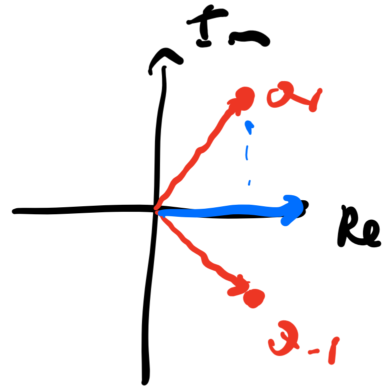


Ex : $\cos(\omega t + \phi) = \dfrac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}$

$$= \left(\frac{e^{j\phi}}{2}\right)\cdot e^{j\omega t} + \left(\frac{e^{-j\phi}}{2}\right)\cdot e^{-j\omega t}$$

↳ $a_1 = \frac{1}{2}\cdot e^{j\phi}$ $\quad |a_1| = \frac{1}{2} \quad \theta_{a_k} = \phi$

↳ $a_{-1} = \frac{1}{2}\cdot e^{-j\phi}$ $\quad |a_{-1}| = \frac{1}{2} \quad \theta_{a_k} = -\phi$



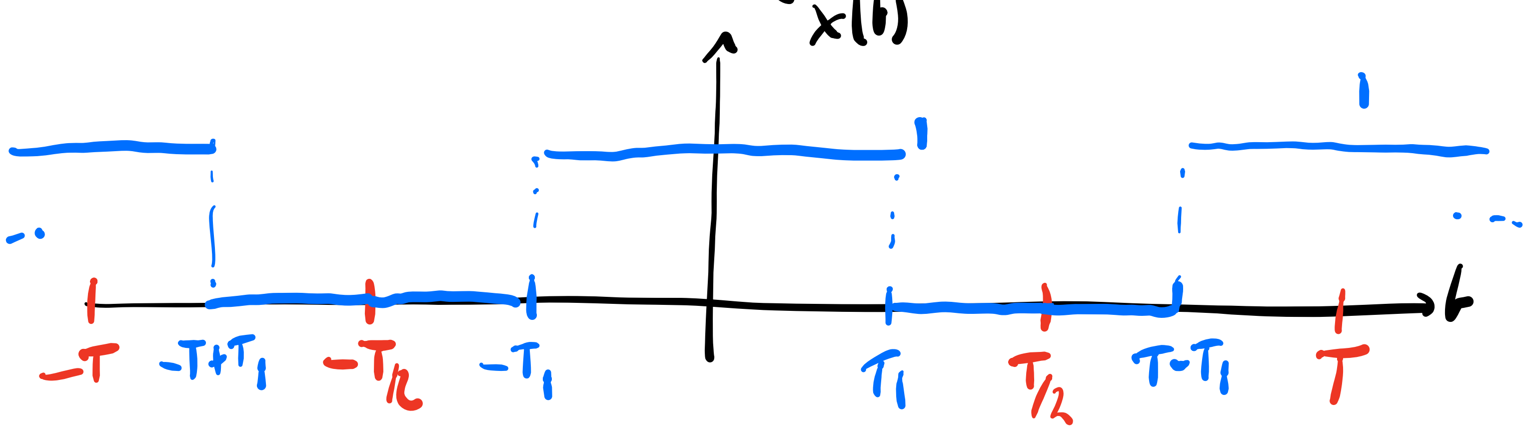
LA MAJORITÉ DES SIGNAUX SONT APÉRIODIQUES

→ EXTENSION POSSIBLE ?

↳ SÉRIES DE FOURIER D'UN SIGNAL RECTANGULAIRE PÉRIODIQUE DE PÉRIODE T.

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 \leq |t| \leq \frac{T}{2} \end{cases}$$

PÉRIODIQUE DE PÉRIODE T



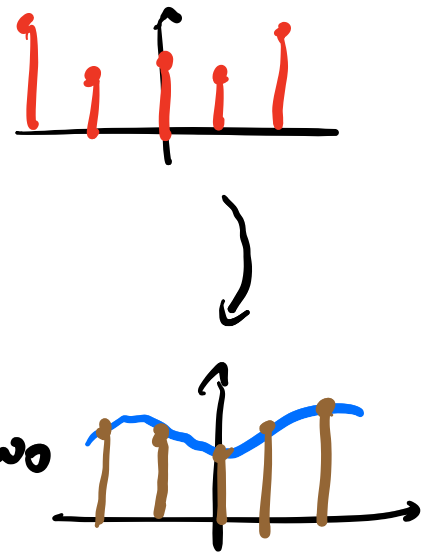
↳ $a_k = \frac{1}{T}\int_T x(t).e^{-jk\omega_0 t}\,dt \quad \Big| \quad \omega_0 = \frac{2\pi}{T}$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

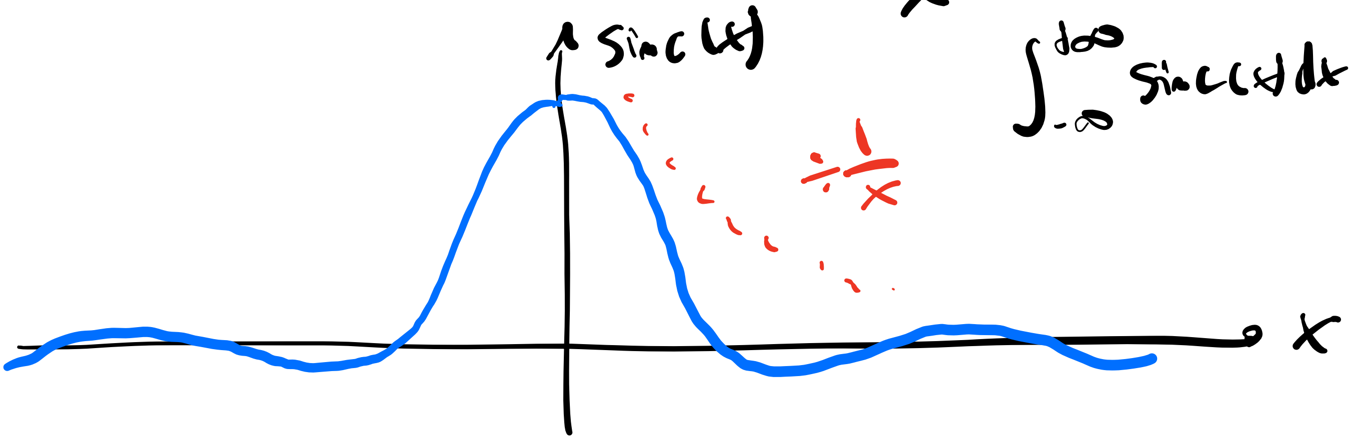
→ REPRESENTATION GRAPHIQUE.

$$T.a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0}$$

$$= \underbrace{T_1\,\frac{2\sin(\omega T_1)}{\omega T_1}}_{\text{ENVELOPPE}} \Bigg|_{\omega = k\omega_0}$$



Sinus cardinal: $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$



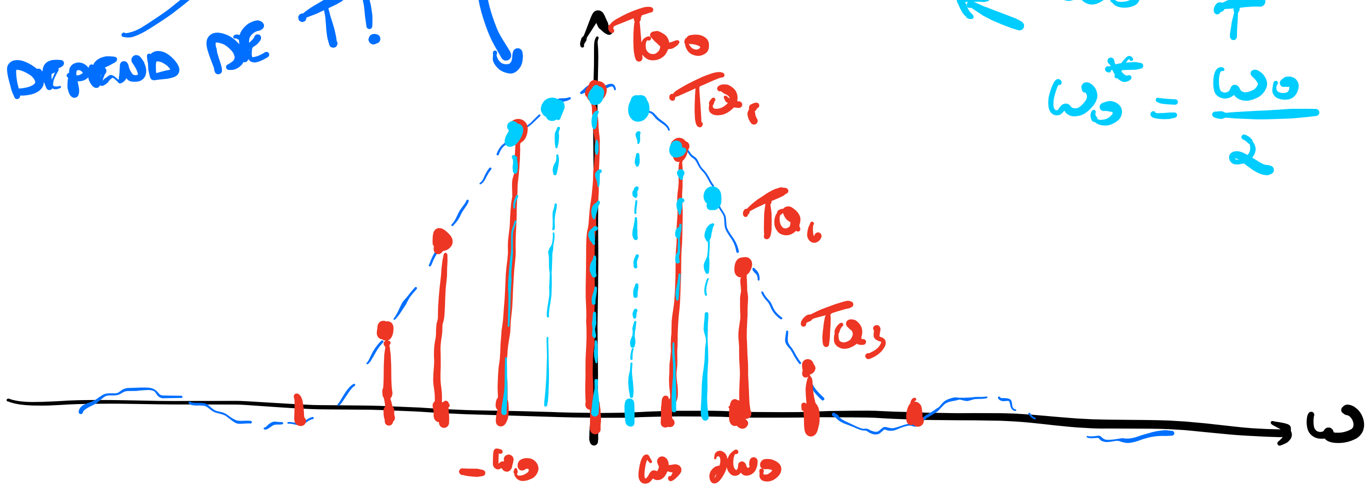
$$\int_{-\infty}^{+\infty} \mathrm{sinc}(x)\,dx = \pi$$

$$T \cdot a_k = \left. 2 \cdot T_1 \, \mathrm{Sinc}(\omega T_1) \right|_{\omega = k\omega_0}$$

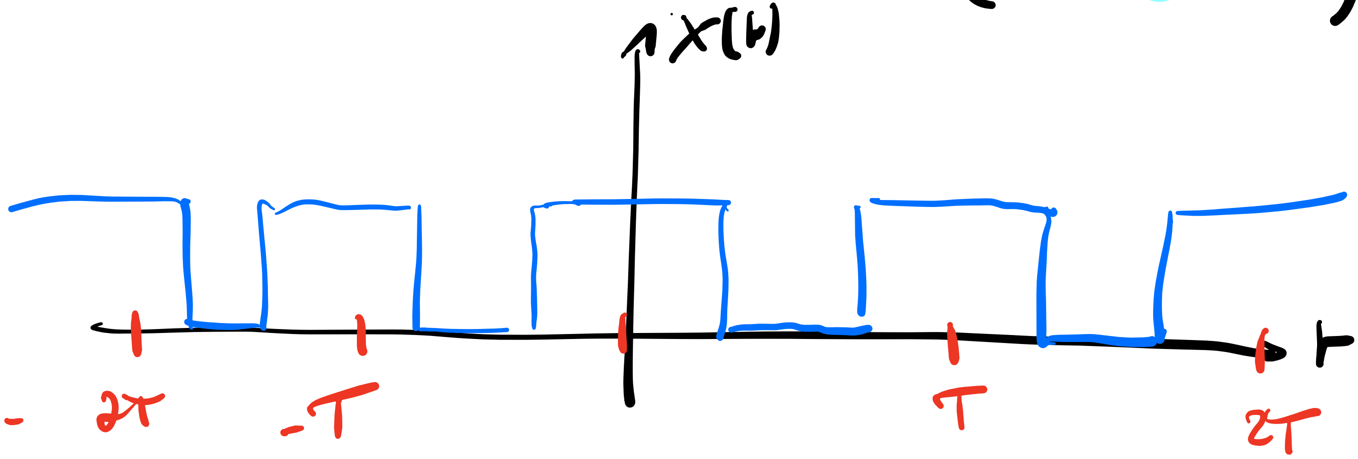
NE DEPEND DE T!

$\omega_0 = \dfrac{2\pi}{T}$

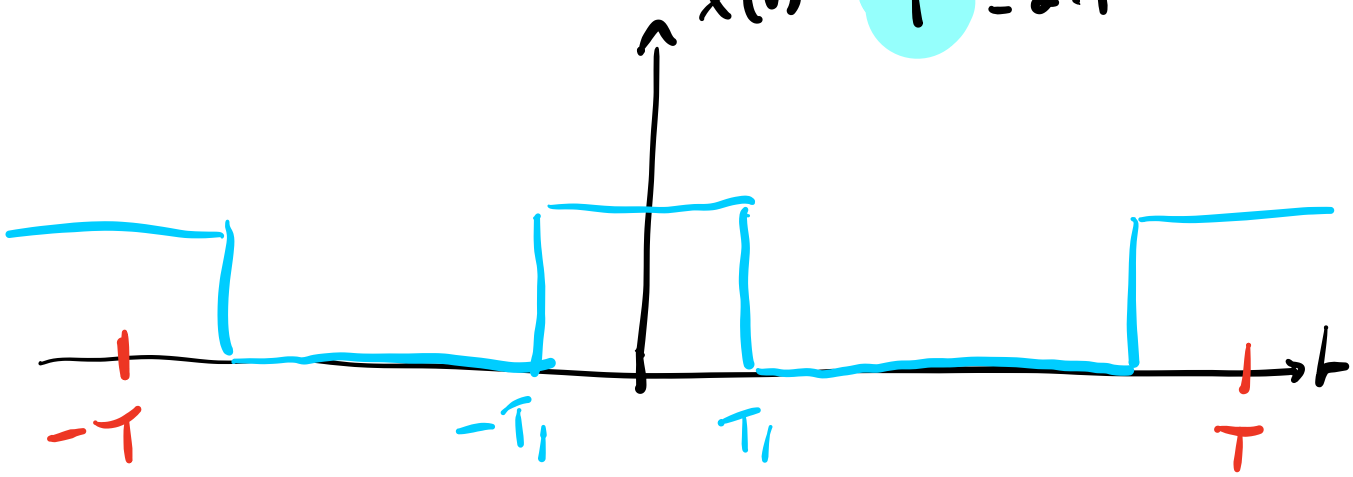
$\omega_0^* = \dfrac{\omega_0}{2}$



↳ QUE SE PASSE-T-IL SI $T \uparrow$ (Ex $T^* = 2T$)

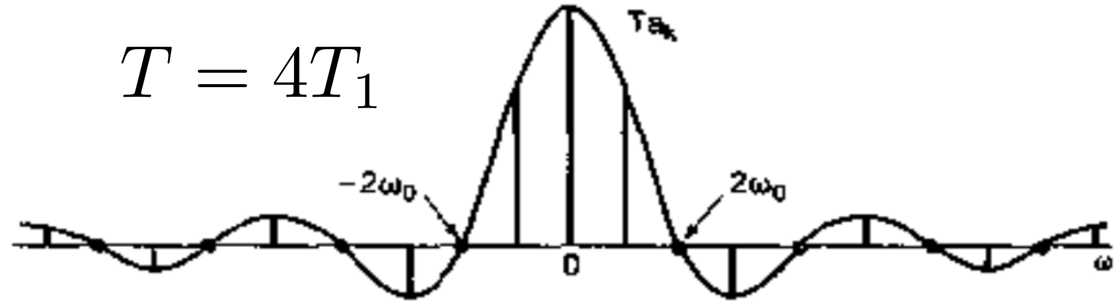


$x(t)$ $T^* = 2 \cdot T$



$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}\bigg|_{\omega = k\omega_0}$$

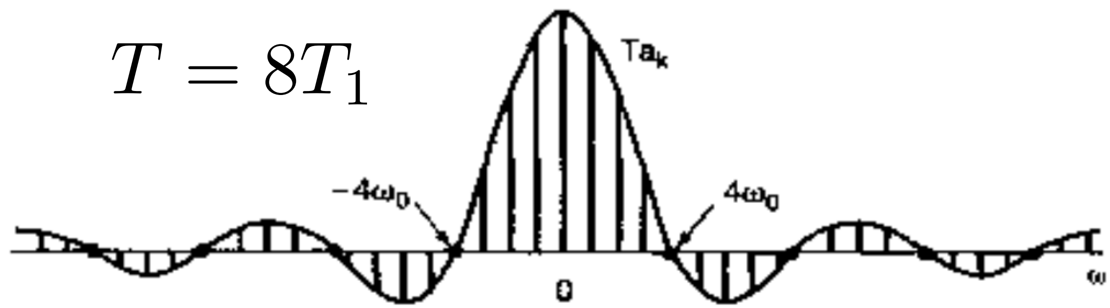
$T = 4T_1$



(a)

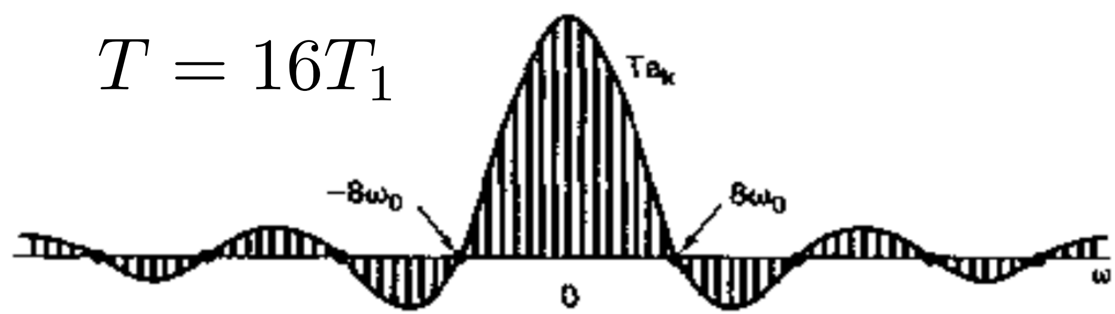
$T \nearrow$

$T = 8T_1$



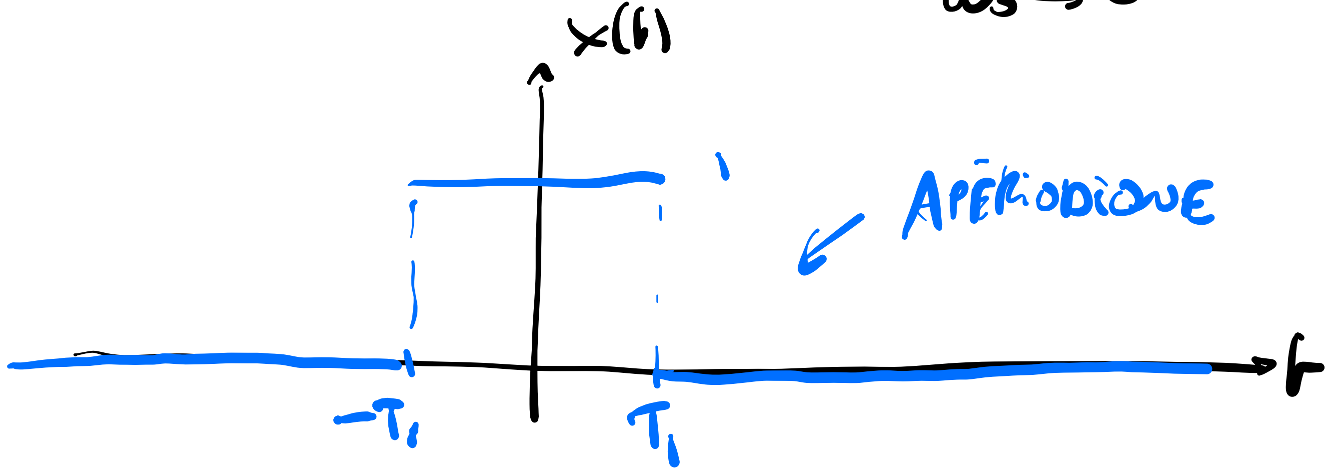
(b)

$T = 16T_1$



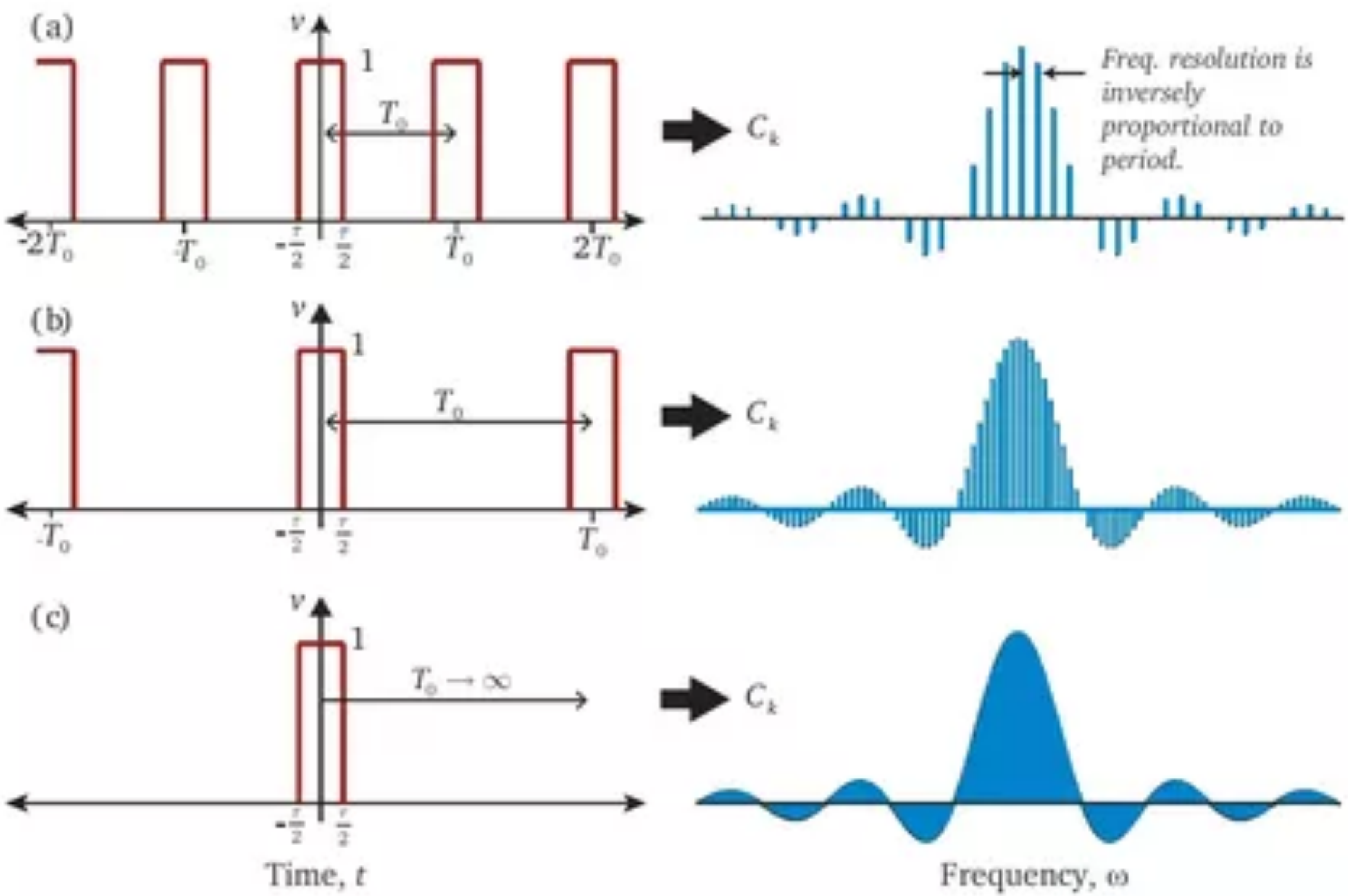
↳ SIGNAL APÉRIODIQUE ? $\lim T \to +\infty$

$\omega_0 \to 0$

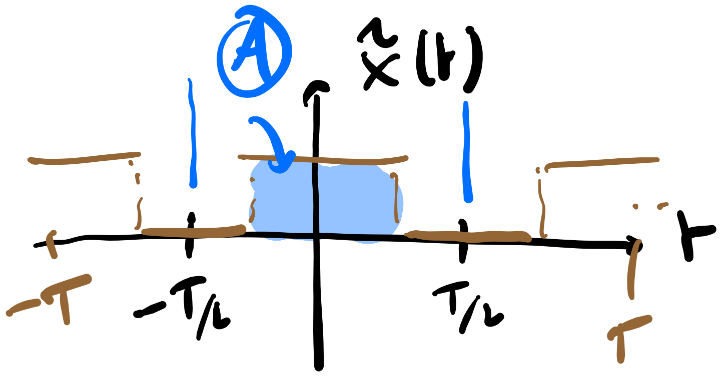


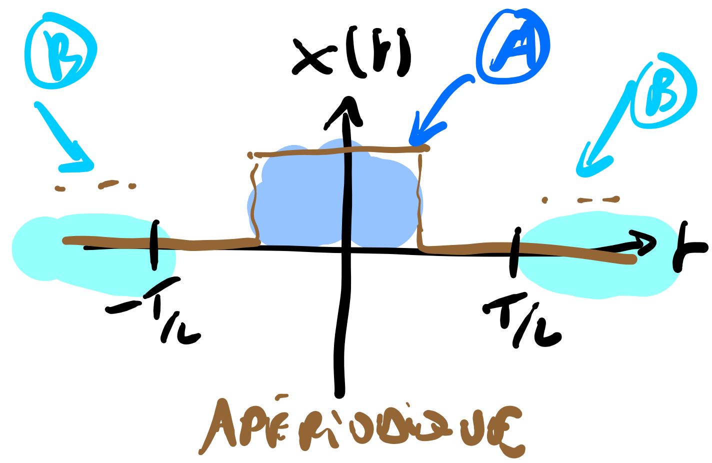
↳ ω_0 DEVIENT INFINITÉSIMALE

→ "TOUTE LES FRÉQUENCES DEVIENNENT DES HARMONIQUES"


(a)
v
1
T_0
C_k
Freq. resolution is inversely proportional to period.
$-2T_0$
$-T_0$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
T_0
$2T_0$
(b)
v
1
T_0
C_k
$-T_0$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
T_0
(c)
v
1
$T_0 \to \infty$
C_k
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
Time, t
Frequency, ω

DÉRIVATION MATHÉMATIQUE DE LA REPRÉSENTATION FRÉQUENTIELLE D'UN SIGNAL APÉRIODIQUE





$$\hookrightarrow \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{-jk\omega_0 t} \quad , \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) \cdot e^{-jk\omega_0 t}\, dt$$

A

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_0 t}\, dt$$

B

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) \cdot e^{-jk\omega_0 t}\, dt$$

ENVELOPPE

$$= \frac{1}{T}\underbrace{\int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt}\Bigg|_{\omega = k\omega_0}$$

$$= \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}$$

$\hookrightarrow X(j\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt$ ENVELOPPE DES T.a_k

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) \cdot e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} \frac{\omega_0}{2\pi} X(jk\omega_0) \cdot e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) \cdot e^{jk\omega_0 t} \cdot \omega_0$$

$\hookrightarrow x(t)$? $\lim_{T \to +\infty} \tilde{x}(t)$

$\omega_0 \to d\omega$

$k \cdot \omega_0 \to \omega$

$\Sigma \to \int$

$$\to x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

↳ TRANSFORMÉE DE FOURIER D'UN SIGNAL CONTINU $x(t)$

$$\mathcal{F} \left| \; X(j\omega) = \int_{-\infty}^{+\infty} x(t).e^{-j\omega t}\,dt \right.$$

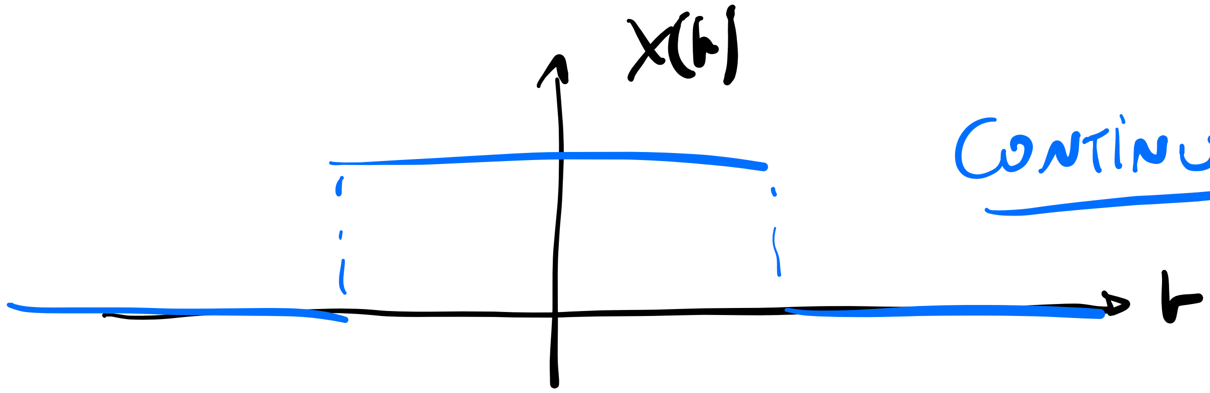
APÉRIODIQUE

CONTINU

$$\mathcal{F}^{-1} \left| \; x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega).e^{j\omega t}\,d\omega \right.$$

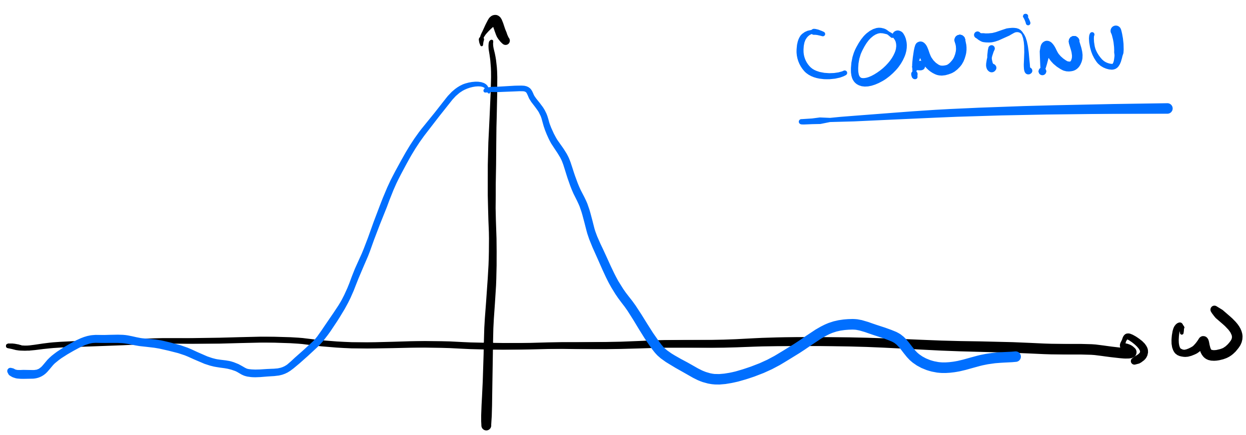
APÉRIODIQUE

CONTINU

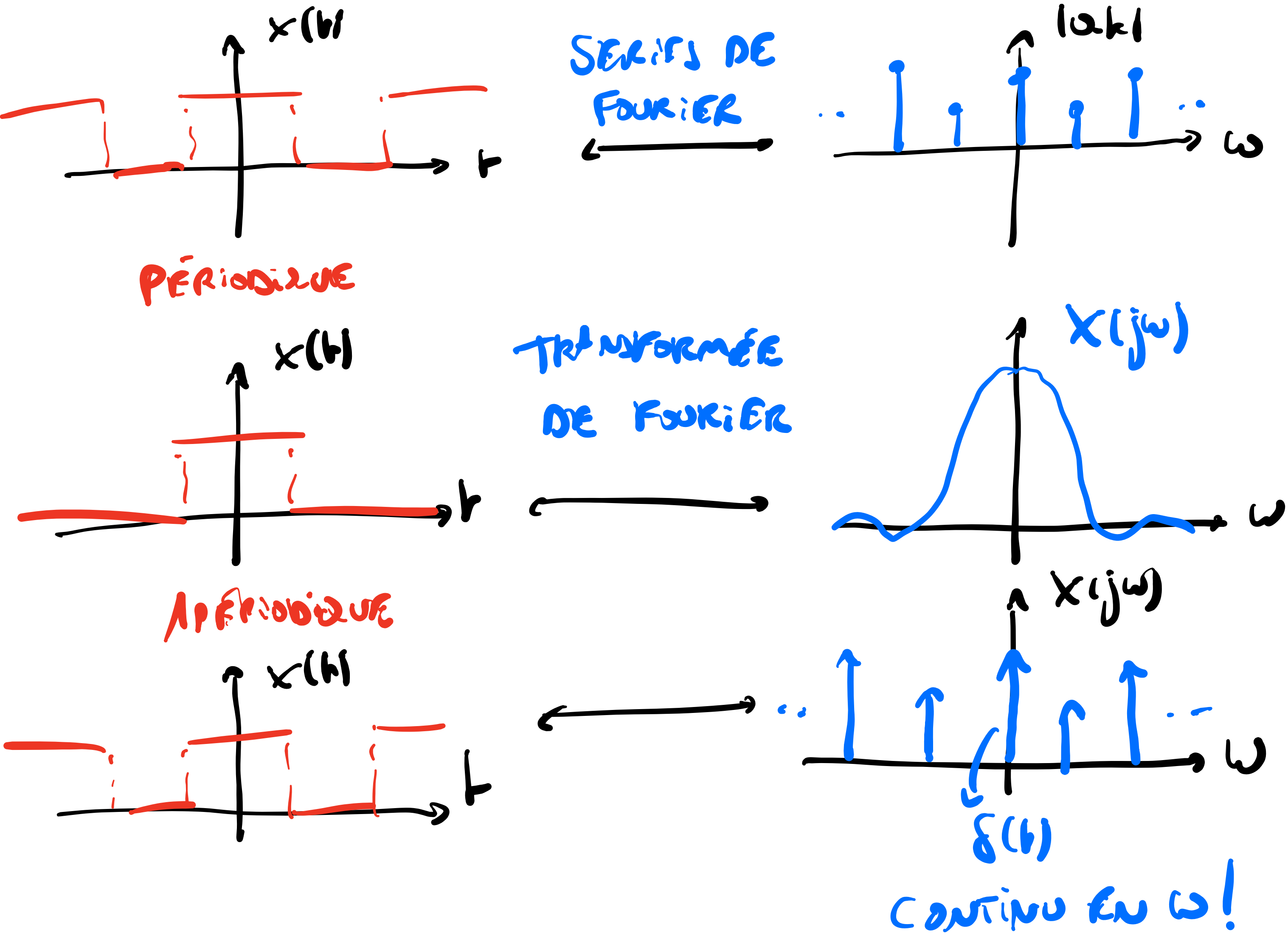


CONTINU

$\mathcal{F}^{-1}$ $\mathcal{F}$



CONTINU



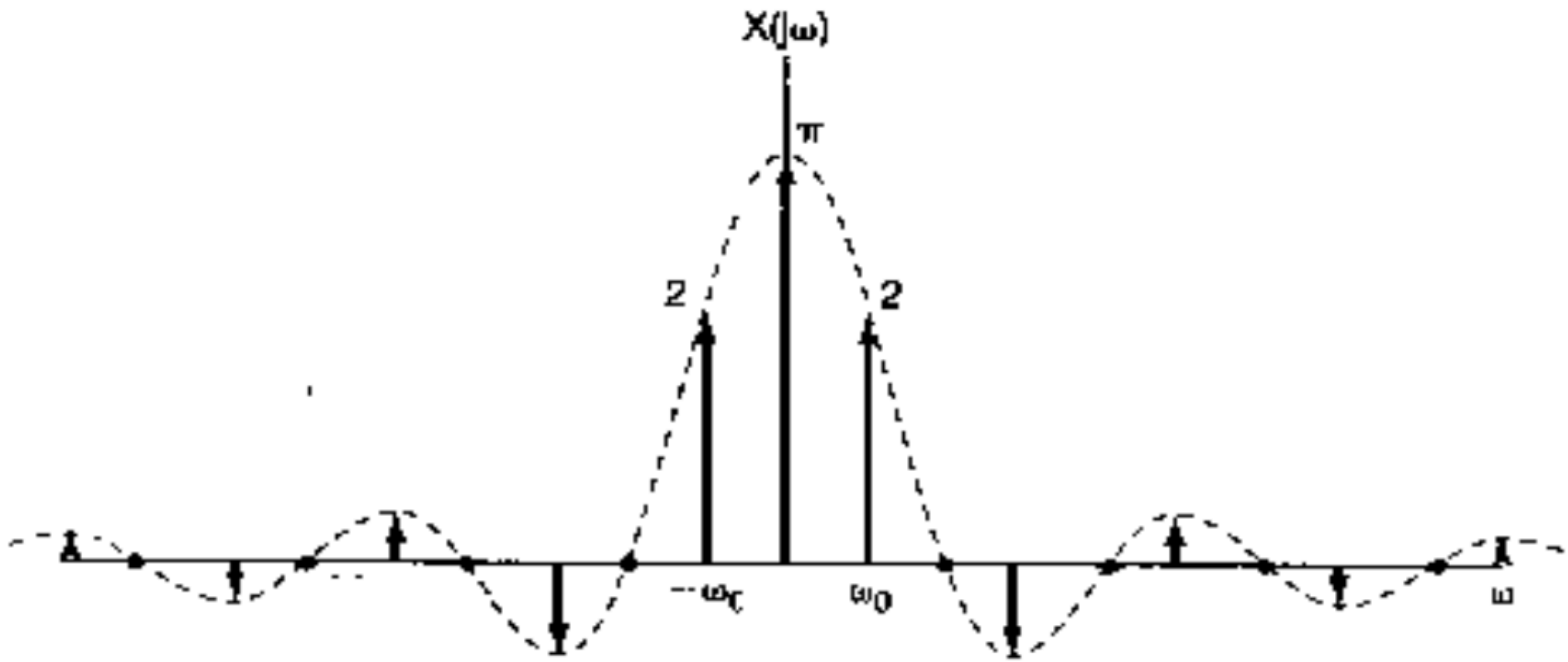
$x(t) \div t$

$X(j\omega) \div \omega$ $[X(\omega)\,?]$

↳ FRÉQUENCE COMPLEXE : $s = \sigma \pm j\omega$

$[X(s)]$ FOURIER : $s = j\omega$ $[\sigma = 0]$

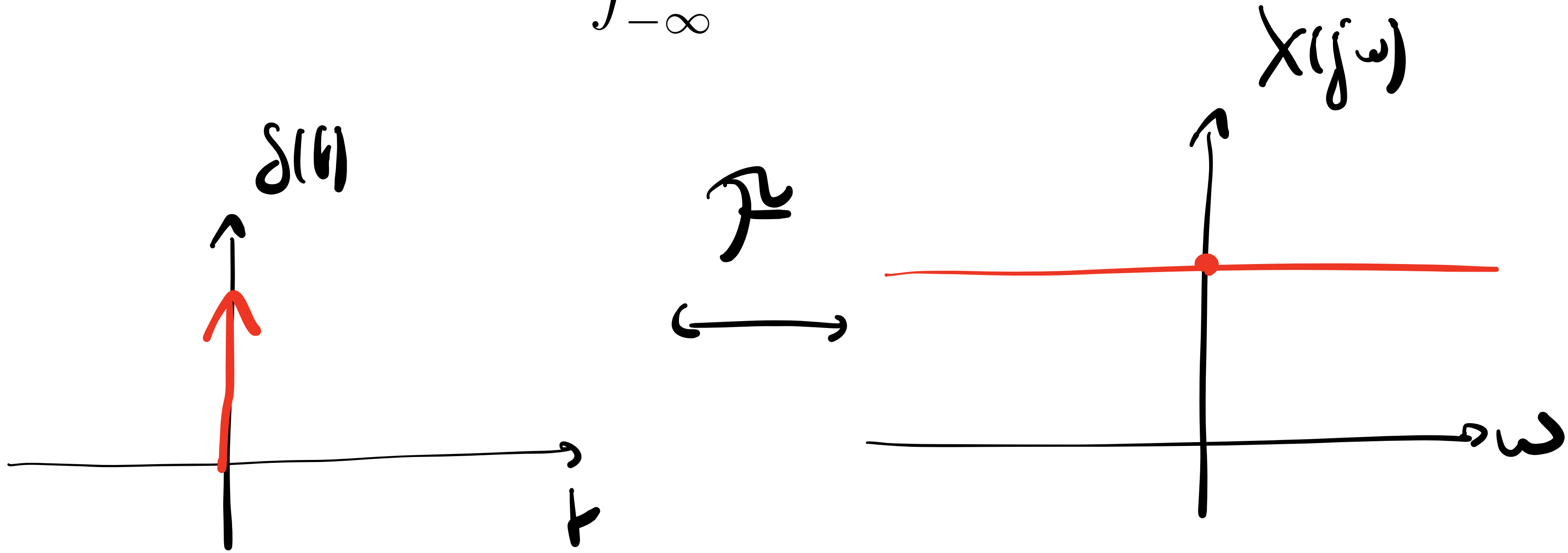
↳ $X(j\omega)$


X(jω)
π
2
2
−ω0
ω0
ω

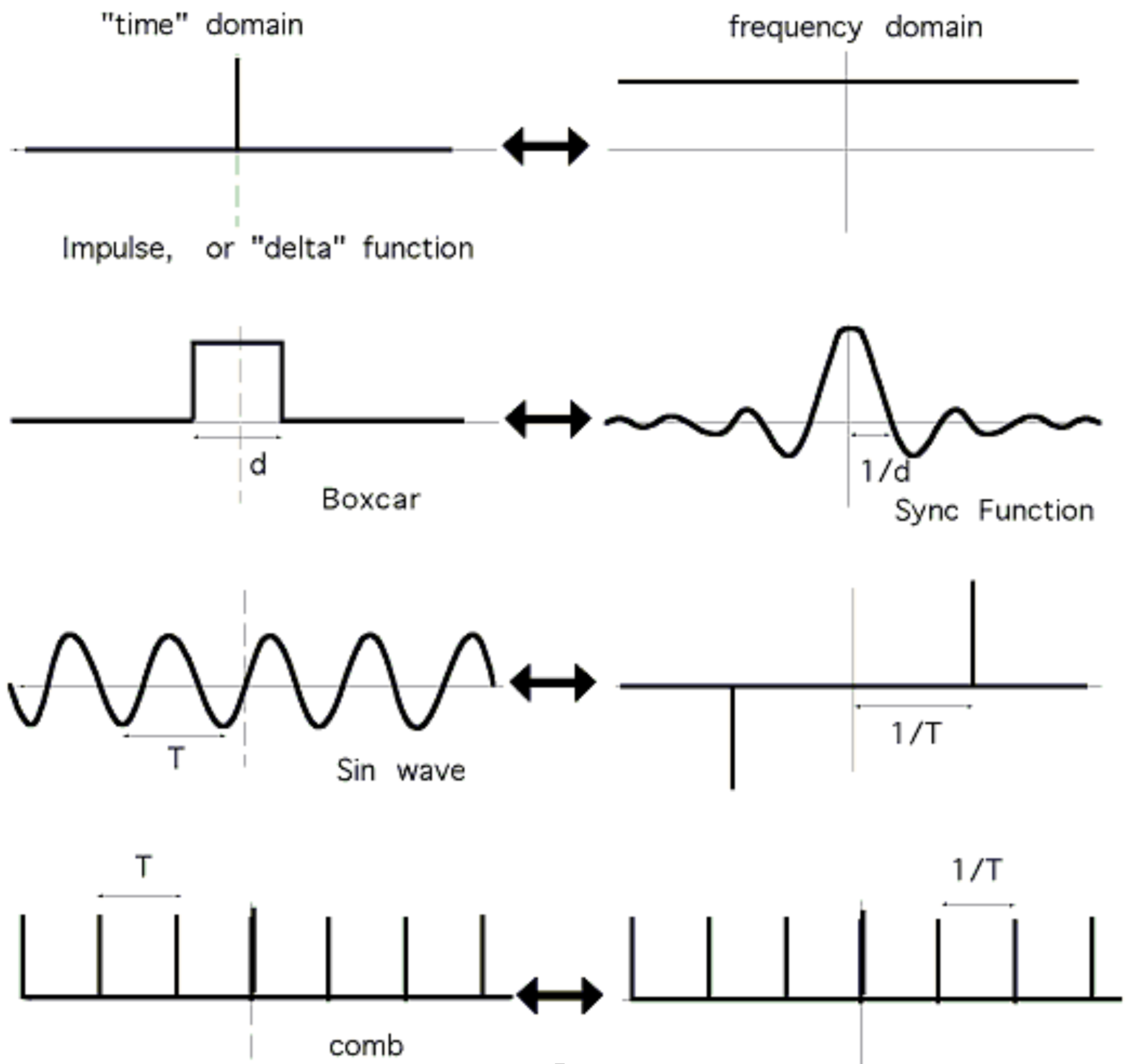
Examples of Fourier transforms

- Fourier transform of Dirac delta function $x(t) = \delta(t)$:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{j\omega t} dt = 1$$



Examples of Fourier transforms



Convergence of Fourier transforms

- The Fourier transform remain valid for an extremely broad class of signals of infinite duration. The convergence criteria are similar to the Fourier series:

- There is no energy in the difference between a signal and the reconstruction of a signal from its Fourier representation if the signal has finite energy:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty$$

- The **Dirichlet conditions** ensure that a signal equals its Fourier representation, except at isolated values of time for which the signal is discontinuous (see Fourier series).

Properties of the Fourier transform: linearity

If

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

and

$$y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$$

then

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$$

Properties of the Fourier transform: time shifting

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

It is very useful when taking time delays into account in systems.

Properties of the Fourier transform: differentiation and integration

$$\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Properties of the Fourier transform: time and frequency scaling

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

then

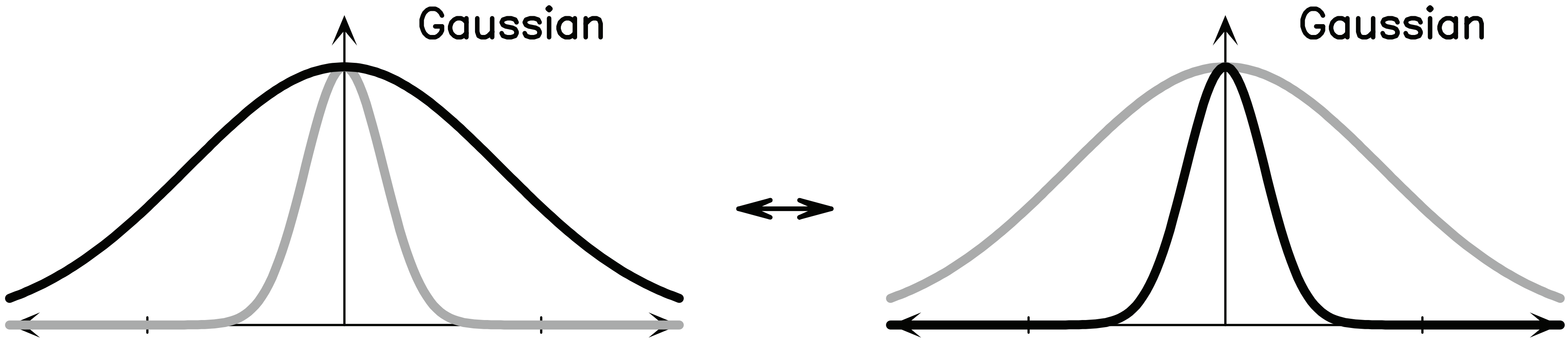
$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$$

Example: playing an audio signal faster sounds higher in frequency.

Properties of the Fourier transform: time and frequency scaling

If $\quad x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad$ then $\quad x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$

A signal that is localized in time is not localized in frequency, and conversely!



Time domain

Frequency domain

Example: incertitude principle in physics.

Properties of the Fourier transform: the convolution property

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = u(t) * h(t)$$

$$\updownarrow \mathcal{F} \qquad \begin{aligned} u(t) &\longleftrightarrow U(j\omega) \\ h(t) &\longleftrightarrow H(j\omega) \\ y(t) &\longleftrightarrow Y(j\omega) \end{aligned}$$

$$U(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = U(j\omega) \cdot H(j\omega)$$

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(j\omega)$$

$$y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

then

$$y(t) = x(t) * h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = X(j\omega)H(j\omega)$$