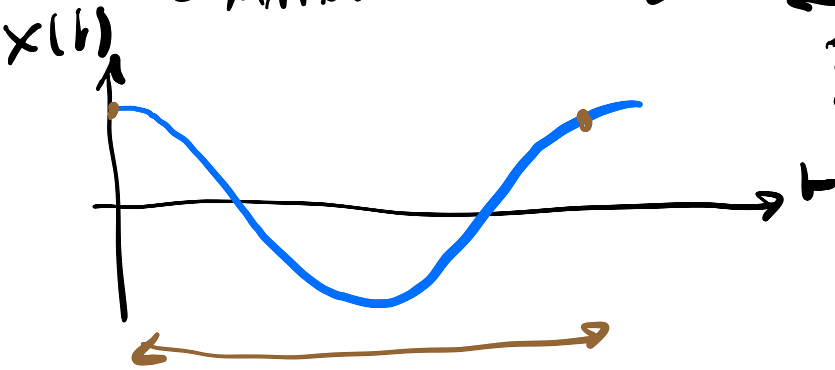
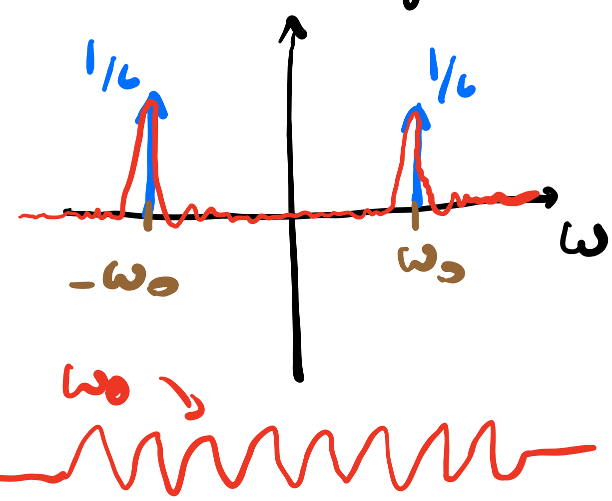
SYST0002 - LECTURE #7 LTI ET DOMAINE FRÉQUENTIEL.

↳ RAPPEL :

DOMAINE TEMPOREL: $\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}$ FREQ. $|X(j\omega)|$



$T = \dfrac{1}{f_0} = \dfrac{2\pi}{\omega_0}$



↳ . TRANSFORMÉE DE FOURIER

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt$$

. TRANSFORMÉE DE FOURIER INVERSE

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \, d\omega$$

↳ ENTRÉE - SORTIE $\quad u \to \boxed{S(\cdot)} \to y = S(u)$

LTI :

$$u^* = \sum_i \alpha_i u_i \xleftrightarrow{LTI} y^* = \sum_i \alpha_i y_i$$

① $u_i(t) = \delta(t) \longleftrightarrow y_i(t) = h(t) \Longleftrightarrow \underline{h(t)}$.

② $x_i(t) = e^{j\omega_i t} \longrightarrow \boxed{LTI} \longrightarrow y_i(t) = ?$

$$\hookrightarrow y_i(t) = x_i(t) * h(t) = h(t) * x_i(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, x_i(t-\tau)\, d\tau$$

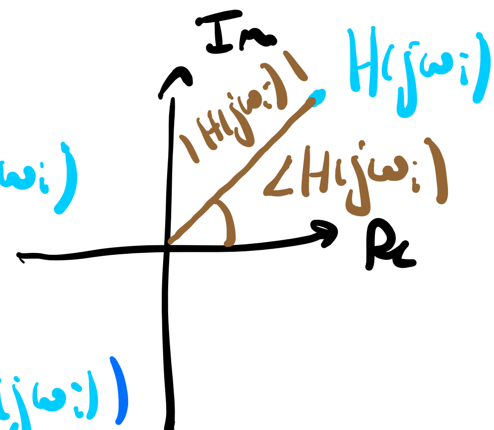
$$= \int_{-\infty}^{+\infty} h(\tau) \cdot e^{j\omega_i (t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) \cdot e^{j\omega_i t} \cdot e^{-j\omega_i \tau}\, d\tau$$

$$= e^{j\omega_i t} \cdot \underbrace{\int_{-\infty}^{+\infty} h(\tau) \cdot e^{-j\omega_i \tau}\, d\tau}_{\text{INDÉPENDANT DE } t\,!}$$

$$= e^{j\omega_i t} \cdot H(j\omega_i)$$

$$= e^{j\omega_i t} \cdot |H(j\omega_i)| \cdot e^{j\angle H(j\omega_i)}$$



$$= \underbrace{|H(j\omega_i)|}_{\text{AMPLITUDE}} \cdot e^{j(\omega_i t + \underbrace{\angle H(j\omega_i)}_{\text{PHASE}})}$$

FREQ IDENTIQUE !

$x_i(t) \cdot e^{j\omega_i t}$

↳ $u^*(t) = \sum_i a_i \, e^{j\omega_i t}$

↳ $\to \boxed{H(j\omega)} \to y^*(t) = \sum_i a_i \cdot e^{j\omega_i t} \cdot H(j\omega_i)$

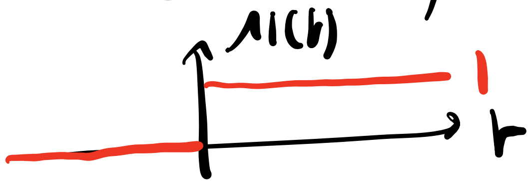
↳ $u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(j\omega) \cdot e^{j\omega t} \, d\omega$

↳ $\to \boxed{H(j\omega)} \to y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(j\omega) \cdot e^{j\omega t} \cdot H(j\omega) \, d\omega$

⚠ $H(j\omega) = \underbrace{\int_{-\infty}^{+\infty} h(t) \cdot e^{-j\omega t} \, dt}_{\text{CONVERGENCE !}}$

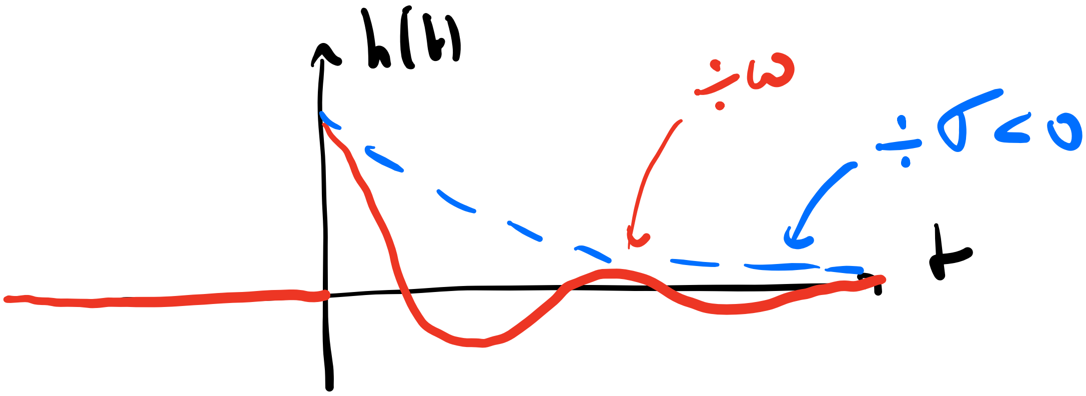
Ex: STABLE, CAUSAL, ORDRE I $[\dot{x} = \lambda x + u]$ $\lambda < 0$

$\hookrightarrow h(t) = C_1 e^{\lambda t} \mathbb{1}(t)$



$\longrightarrow$ FOURIER CONVERGE.

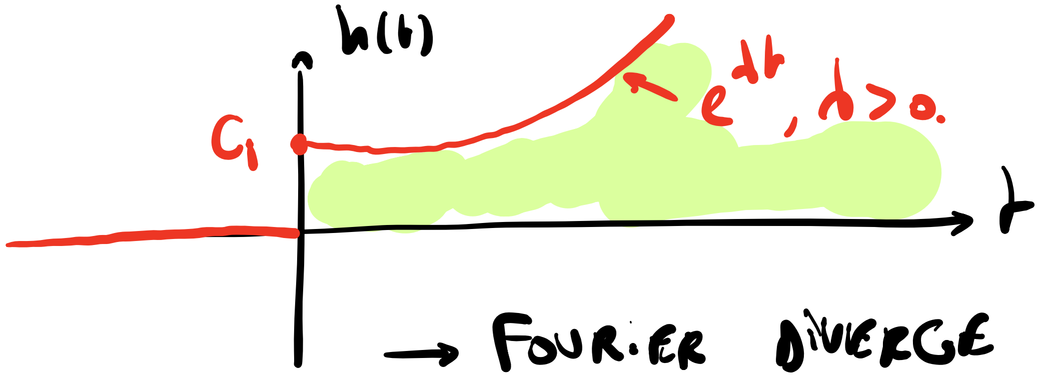
Ex: STABLE, CAUSAL, ORDRE 2 $[\lambda_{1,2} = \sigma \pm j\omega \,;\ \sigma < 0]$



$\longrightarrow$ FOURIER CONVERGE.

EX : INSTABLE, CAUSAL, ORDRE 1

$$h(t) = C_1 \cdot e^{\lambda t} \cdot 1\!\!1(t) \quad , \quad \lambda > 0$$



→ FOURIER DIVERGE

↳ N'EXISTE PAS !

<u>IDÉE</u> : RENDRE $h(t)$ "INTÉGRABLE"

↳ $h(t) \cdot e^{-\sigma t}$ t.q. $\int_{-\infty}^{+\infty} h(t) \cdot e^{-\sigma t} \, dt$

$\sigma \in \mathbb{R}$ CONVERGE.

→ $h(t) = e^{\lambda t} 1\!\!1(t)$, $\lambda > 0$. [CAUSAL, INSTABLE]

$$↳ H(\cdot) = \int_{-\infty}^{+\infty} \underbrace{1\!\!1(t) \cdot e^{\lambda t} \cdot e^{-\sigma t}}_{h(t) \cdot e^{-\sigma t}} \cdot \underbrace{e^{-j\omega t}}_{\text{FOURIER}} \, dt$$

$$= \int_{-\infty}^{+\infty} 1\!\!1(t) \cdot \underbrace{e^{(\lambda - \sigma) t}}_{} \cdot \underbrace{e^{-j\omega t}}_{} \, dt$$

↳ $H(\cdot)$ EXISTE SI $\lambda - \sigma < 0$; $\sigma > \lambda$.

$$\rightarrow H(\cdot) = \int_{-\infty}^{+\infty} \underbrace{\mathbb{1}(t)\cdot e^{\sigma t}}_{h(t)} \cdot e^{-\sigma t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} h(t)\cdot e^{-(\underbrace{\sigma + j\omega}_{})t}\, dt$$

$\sigma + j\omega \longleftrightarrow s = \sigma + j\omega$

FRÉQUENCE COMPLEXE.

$$H(s) = \int_{-\infty}^{+\infty} h(t)\cdot e^{-st}\, dt$$

$\rightarrow$ TRANSFORMÉE DE LAPLACE DE $h(t)$.

TRANSFORMÉE DE LAPLACE (BILATÉRALE)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\underbrace{\sigma + j\omega}_{s}) . e^{\overbrace{(\sigma+j\omega)}^{s} t} \, d\omega \quad \in \mathbb{R}$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds \quad \in \mathbb{C}$$

$$X(s) = \int_{-\infty}^{+\infty} x(t) . e^{-st} \, dt \, , \quad s = \sigma \pm j\omega.$$

↳ Si $\sigma = 0.$ → TRANSFORMÉE DE FOURIER

$$X(s) = \int_{-\infty}^{+\infty} x(t) . e^{-(\sigma + j\omega)t} \, dt$$

$$= \int_{-\infty}^{+\infty} x(t) . e^{-j\omega t} \, dt = X(j\omega)$$

$$X(j\omega) = X(s) \Big|_{s = j\omega, \; \underline{\sigma = 0.}}$$

→ FOURIER : CONVERGE OU NON.

→ LAPLACE : CONVERGE POUR CERTAINES VALEURS DE $\sigma \in \mathbb{R}$.

→ RÉGION DE CONVERGENCE (ROC)

Ex: $h(t) = \mathbb{1}(t) \cdot e^{-at}$, $a > 0$

$\hookrightarrow H(s) = \int_{-\infty}^{+\infty} h(t) \cdot e^{-st} \, dt$

$= \int_{-\infty}^{+\infty} \mathbb{1}(t) \cdot e^{-at} \cdot e^{-st} \, dt$

$= \int_{0}^{+\infty} e^{-at} \cdot e^{-st} \, dt$

$= \int_{0}^{+\infty} e^{-(a+s)t} \, dt$

$= -\frac{1}{a+s} \left[e^{-(a+s)t} \right]_0^{+\infty}$

$= -\frac{1}{a+s} \left[\underbrace{e^{-(a+s) \cdot +\infty}}_{=0} - \underbrace{e^{-(a+s) \cdot 0}}_{1} \right]$

si $\Re\{a+s\} > 0$

$\hookrightarrow$ ① $\Re\{a+s\} > 0 \longrightarrow H(s) = \frac{1}{a+s}$

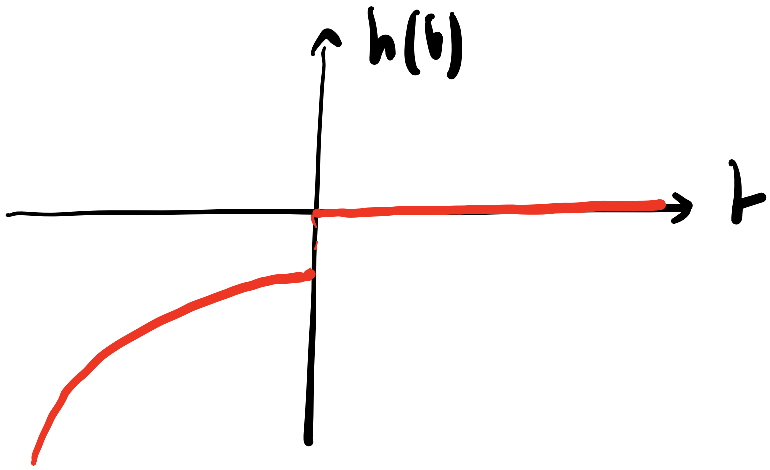
$\hookrightarrow$ ② $\Re\{a+s\} < 0 \longrightarrow$ DIVERGE !

$\rightarrow H(s) = \frac{1}{s+a}$; $\Re\{s\} = \sigma > -a$

TRANSFORMÉE

RÉGION DE CONVERGENCE.

Ex: $h'(t) = -\mathbb{1}(-t) \cdot e^{-at}$, $a > 0$

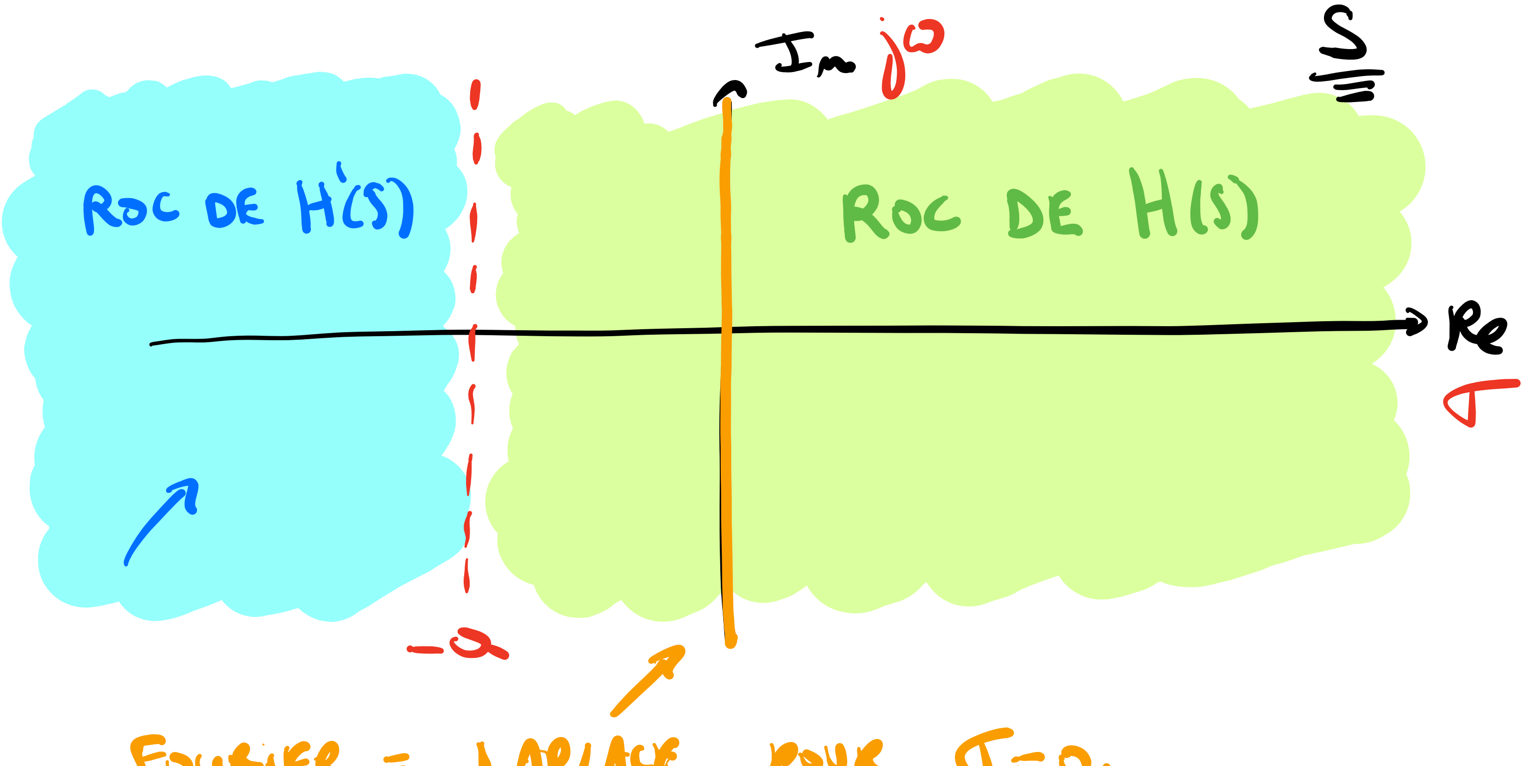


$H'(s) = \dfrac{1}{s+a}$

$\underline{Re\{s\} = \sigma < -a}$

↳ RÉGION DE CONVERGENCE (ROC)

⟶ PLAN COMPLEXE.



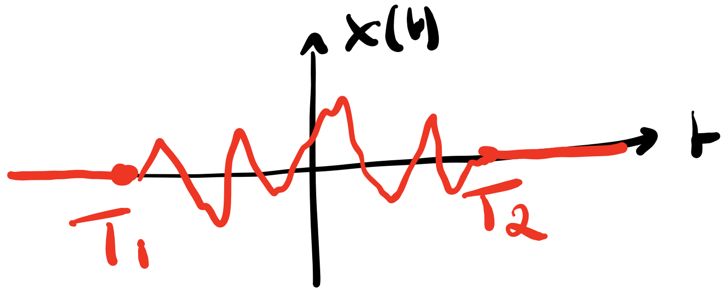
FOURIER ≡ LAPLACE POUR $\sigma = 0$.

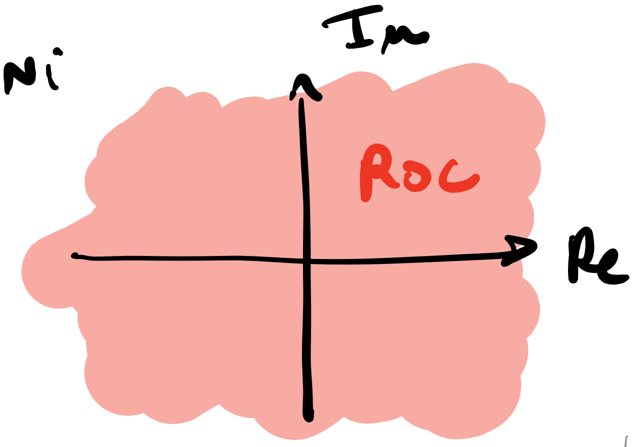
↳ FOURIER EXISTE POUR $h(t)$.

↳ FOURIER N'EXISTE PAS POUR $h'(t)$

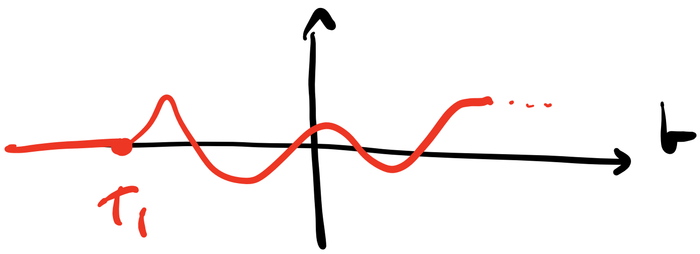
ROC DE $X(s)$: 3 CAS.

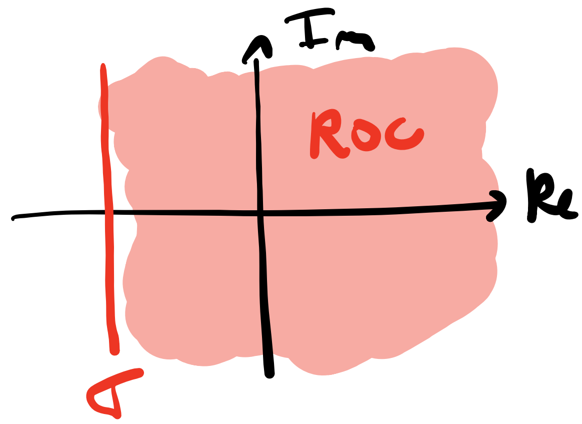
① $x(t)$ EST À SUPPORT FINI



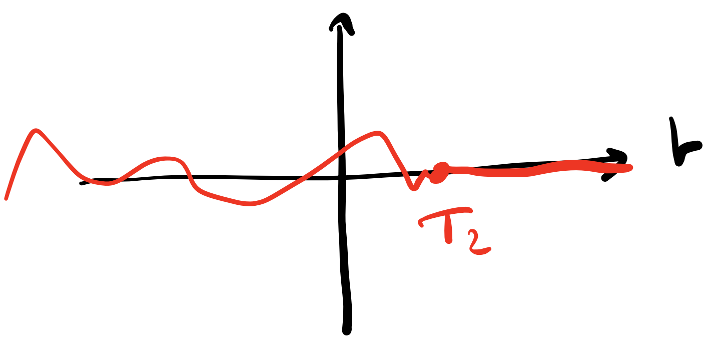


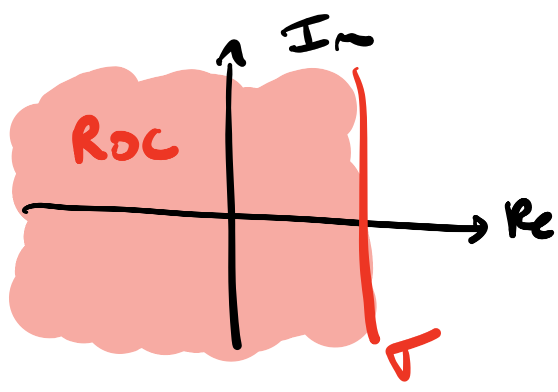
② $x(t)$ EST À SUPPORT FINI À GAUCHE $\cdot e^{-\sigma t}$





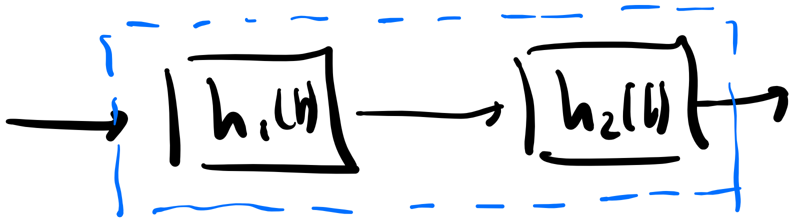
③ $x(t)$ EST À SUPPORT FINI À DROITE



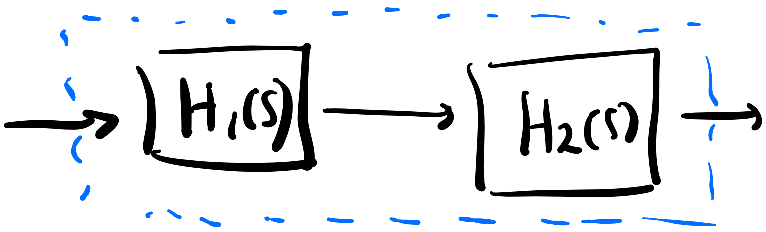


↳ PROPRIÉTÉS PRINCIPALES DE $\mathcal{L}$. (LAPLACE).

(A) $(x_1 * x_2)(t) \xleftrightarrow{\mathcal{L}} X_1(s) \cdot X_2(s)$



$h(t) = h_1(t) * h_2(t)$



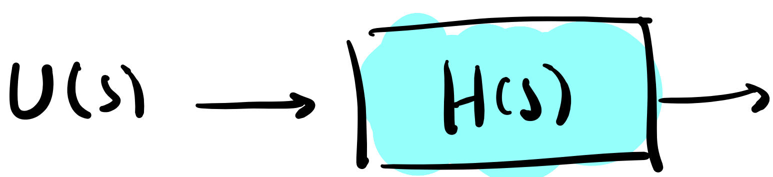
$H(s) = H_1(s) \cdot H_2(s)$

(B) $\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} s \cdot X(s)$

$\frac{d^2x(t)}{dt^2} \xleftrightarrow{\mathcal{L}} s^2 X(s)$

$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s)$

↳ PASSER EN FRÉQUENTIEL PERMET DE NE PLUS TRAITER AVEC DES ODE'S.



$Y(s) = H(s) \cdot U(s)$

FONCTION DE TRANSFERT