TP8 : Transformée de Fourier

Séries de Fourier

Signal périodique ⟶ représentation discrète

$$\begin{cases} x(t) = \sum_{k\in\mathbb{Z}} \hat{x}_k \, e^{j\omega k t} & (1) \\ \hat{x}_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega k t}\, dt & (2) \end{cases}$$

Transformée de Fourier

Pour $T \to \infty$: $\Delta\ell \to d\ell$, $\ell_n \to \ell$

$$\sum_{k=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty}, \quad \int_T \to \int_{-\infty}^{+\infty}$$

$$\Leftrightarrow \begin{cases} x(t) = \int_{-\infty}^{+\infty} \hat{x}_\ell \, e^{j2\pi\ell t}\, d\ell \\ \hat{x}_\ell = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi \ell t}\, dt \end{cases}$$

⚠ Expressions intégrables !

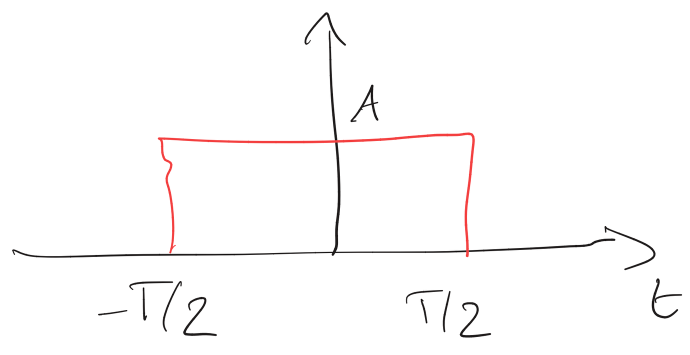
En notant $\ell = f$:

$$\boxed{\begin{cases} x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df \\ X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \end{cases}}$$

Pulsation $\omega = 2\pi f$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega \\ X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

Exercices



1) $x(t) = \begin{cases} A & \text{si } |t| < T/2 \\ 0 & \text{sinon} \end{cases}$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \\ &= \int_{-T/2}^{T/2} A\, e^{-j\omega t}\, dt \\ &= -\frac{A}{j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2} \\ &= -\frac{A}{j\omega}\left(e^{-j\omega T/2} - e^{j\omega T/2}\right) \\ &= \frac{2A}{\omega}\sin\left(\omega \frac{T}{2}\right) \\ &= \frac{A}{\pi f}\sin(\pi f T) \\ &= \boxed{AT\,\mathrm{sinc}(fT)} \end{aligned}$$

$\mathrm{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$

3)

a) $x(t) = \cos(\omega_0 t)$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} \cos(\omega_0 t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{+\infty}\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\, dt \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{j(\omega_0 - \omega)t}\, dt + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-j(\omega_0 + \omega)t}\, dt \end{aligned}$$

⟶ ne mène à rien

Décalage temporel : $\mathcal{F}\{\delta(t)\} = 1 \quad \mathcal{F}\{\delta(t-t_0)\} = e^{-j\omega t_0}$

Décalage fréquentiel : $\mathcal{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$

$$X(j\omega) = \mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\mathcal{F}\{e^{j\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-j\omega_0 t}\}$$

$$\boxed{= \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

b)